Chapter 5

Hardy Spaces.

5.1 Stationary Gaussian Processes.

Consider a collection$\{X_j : -\infty < j < \infty\}$ of random variables such that the joint distribution of any finite collection is a multivariate Gaussian distribution with mean 0 and covariance $E[X_i X_j] = \rho(i-j)$. The sequence $\rho(n)$ is a positive definite sequence and by a theorem of Bochner it has representations as

$$\rho(n) = \int_0^{2\pi} e^{i\,n\,\theta} \mu(d\theta)$$

where μ is a nonnegative measure, on the boundary C of the unit disc D. It is referred to as the spectral measure. If P refers to the measure on the space Ω of sequences $\omega = \{x_j\}$ of such a Gaussian process, it is invariant with respect to the shift $x_i \to x_{i+1}$ and for any finite set of complex numbers $\{a_j\}$

$$E^P[|\sum a_j X_j|^2] = \sum_{j,k} \rho(j-k) a_j \bar{a}_k = \int_0^{2\pi} |\sum a_j e^{i\,j\,\theta}|^2 \mu(d\theta) \tag{5.1}$$

We have the Hilbert space $\mathcal{H}$ which is the span of $\{X_j\}$ in $L_2(P)$ and subspaces $\mathcal{H}_n$ that are spanned by $\{X_j : j \le n\}$ and $\mathcal{H} = \overline{\cup_n \mathcal{H}_n}$. $H_{-\infty} = \cap_n \mathcal{H}_n$.

The following are natural questions.

1. Is $\mathcal{H}_{-\infty} = \{0\}$?

2. Is $\mathcal{H}_\infty = \mathcal{H}_{-\infty}$ or equivalently is $\mathcal{H}_n = \mathcal{H}_{n+1}$?

3. What is the projection $\widehat{X}_{n+1}$ of X_{n+1} on $\mathcal{H}_n$? What is the prediction error $\sigma^2 = E^P[|X_{n+1} - \widehat{X}_{n+1}|^2]$?

From equation 5.1 we can rephrase the questions in terms of the spectral measure μ and the spans $\mathcal{H}_n$ of the trigonometric polynomials $\{e^{i\,j\,\theta} : j \le n\}$. We look at the Hilbert space $L_2(C, \mu)$ and the span H_n of $\{e^{i\,j,\theta} : j \le n\}$ and the questions are easily translated to this context. The fact that the distributions are Gaussian plays an important role and allows to limit ourselves to linear combinations of the random variables.

5.2 Hardy Spaces.

For $0 < p < \infty$, the Hardy Space $\mathcal{H}_p$ in the unit disc D with boundary $S = \partial D$ consists of functions $u(z)$ that are analytic in the disc $\{z : |z| < 1\}$, that satisfy (with $z = re^{i\theta}$),

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta < \infty \tag{5.2}$$

Lemma 5.1. *We have the Poisson representation formula that is valid for* $1 \ge r' > r \ge 0$

$$u(re^{i\,\theta}) = \frac{r'^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(r'e^{i\,(\theta-\varphi)})}{r'^2 - 2rr'\cos\varphi + r^2} d\varphi \tag{5.3}$$

The quantity $M(r) = \int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta.$ *is nondecreasing in* r.

Proof. The real and imaginary parts of $u(z)$ are hamonic functions and the Poisson formula is valid. The monotonicity is obvious for $p = 1$ because for $r' > r$,

$$|u(re^{i\,\theta})| \le \frac{r'^2 - r^2}{2\pi} \int_0^{2\pi} \frac{|u(r'e^{i\,(\theta-\varphi)|})}{r'^2 - 2rr'\cos\varphi + r^2} d\varphi$$

and for $p > 1$ it is an application of Hölder's inequality. Actually

$$M(r) = \int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta$$

is monotonic in r for $p > 0$. To see this we note that $g(re^{i\,\theta}) = \log |u(re^{i\,\theta})|$ is subharmonic and therefore, using Jensen's inequality,

$$\begin{aligned}
&\frac{r'^2 - r^2}{2\pi}\int_0^{2\pi}\frac{\exp[pg(r'e^{i(\theta-\varphi)})]}{r'^2 - 2rr'\cos\varphi + r^2}d\varphi \\
&\qquad \geq \exp[p\frac{r'^2 - r^2}{2\pi}\int_0^{2\pi}\frac{g(r'e^{i(\theta-\varphi)})}{r'^2 - 2rr'\cos\varphi + r^2}d\varphi] \\
&\qquad \geq \exp[pg(re^{i\theta})]
\end{aligned}$$

□

Integrating both sides with respect to θ we obtain the inequality.

Theorem 5.2. *If $1 < p < \infty$ and $u(x, y)$ is a Harmonic function in D, satisfying a bound of the form*

$$\sup_{0<r<1}\int_0^{2\pi}|u(re^{i\theta})|^p d\theta \leq C$$

then $\lim_{r\to 1} u(re^{i\theta}) = f(\theta)$ exists in L_p and we obtain the representation

$$u(re^{i\theta}) = \frac{1 - r^2}{2\pi}\int_0^{2\pi}\frac{f(e^{i(\theta-\varphi)})}{1 - 2r\cos\varphi + r^2}d\varphi$$

in terms of the boundary function f on S.

Proof. We can get a weak radial limit f (along a subsequence if necessary) of $u(r'e^{i\theta})$ as $r' \to 1$. In (5.2) we can let $r' \to 1$ keeping r and θ fixed. The Poisson kernel converges strongly in L_q to

$$\frac{1}{2\pi}\frac{1 - r^2}{1 - 2r\cos\varphi + r^2}$$

and we get the representation (5.2) for $u(re^{i\theta})$ (with $r' = 1$) in terms of the boundary function f on S.

Now it is clear that actually

$$\lim_{r\to 1} u(re^{i\theta}) = f(\theta)$$

in L_p. Since we can consider the real and imaginary parts separately, these considerations apply to Hardy functions in $\mathcal{H}_p$ as well. □

The Poisson kernel is harmonic as a function of (r, θ). It is the real part of the function $\frac{1}{2\pi}\frac{1+z}{1-z}$. It is easily seen that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{i\,n\,\theta} = \frac{1}{1-z} + \frac{1}{1-\bar{z}} - 1 = \frac{1-r^2}{1-2r\cos\theta + r^2} = R.P.\left[\frac{1+z}{1-z}\right]$$

and its harmonic conjugate, normalized so that $u(0) = 0$ is given by

$$I.P.\left[\frac{1+z}{1-z}\right] = \sum_{n=-\infty}^{\infty} r^{|n|}e^{i\,n\,\theta}\sigma(n)$$

where $\sigma(n) = \pm 1$ or 0 depending on whether n is positive, negative or 0. It is clear that any function in the Hardy Spaces is essentially determined by the boundary value of its real (or imaginary part) on S. The conjugate part is then determined through the Hilbert transform and to be in the Hardy class $\mathcal{H}_p$, both the real and imaginary parts should be in $L_p(R)$. For $p > 1$, since the Hilbert transform is bounded on L_p, this is essentially just the condition that the real part be in L_p. However, for $p \leq 1$, to be in $\mathcal{H}_p$ both the real and imaginary parts should be in L_p, which is stronger than just requiring that the real part be in L_p.

5.3 Inner and outer functions.

We now prove a factorization theorem for functions $u(z) \in \mathcal{H}_p$ for p in the range $0 < p < \infty$.

A function analytic in the disc D is called an **inner function** if it is bounded by 1 and its boundary value f on S that exists as a radial limit in every L_p, is of modulus 1, i.e $|f| = 1$ a.e. on S.

A function analytic in the disc D which is in H_1 is called an **outer function** if

$$\log|u(re^{i\,\theta})| = \frac{1-r^2}{2\pi}\int \frac{\log|ue^{i\,\phi}|}{1-2r\cos(\theta-\phi)+r^2}d\phi$$

Theorem 5.3. *Let $u(z) \in \mathcal{H}_p$ for some $p \in (0, \infty)$. Then there exists a factorization $u(z) = v(z)F(z)$ of u into two analytic functions v and F on D with the following properties. $|F(z)| \leq 1$ in D and the boundary value $F^*(e^{i\,\theta}) = \lim_{r\to 1} F(re^{i\,\theta})$ that exists in every $L_p(S)$ satisfies $|F^*| = 1$ a.e. on S. Moreover F contains all the zeros of u so that v is zero free in D.*

Proof. Suppose u has a zero at the origin of order k and no other zeros. Then we take $F(z) = z^k$ and we are done. In any case, we can remove the zero, if any, at 0 and are therefore free to assume that $u(z) \neq 0$. Suppose u has a finite number of zeros, $z_1, \ldots, z_n$. For each zero z_j consider $f_{z_j}(z) = \frac{z - z_j}{1 - z\bar{z}_j}$. A simple calculation yields $|z - z_j| = |1 - z\bar{z}_j|$ for $|z| = 1$. Therefore $|f_{z_j}(z)| = 1$ on S and $|f_{z_j}(z)| < 1$ in D. We can write $u(z) = v(z)\Pi_{i=1}^{n} f_{z_j}(z)$. Clearly the factorization $u = Fv$ works with $F(z) = \Pi f_{z_i}(z)$. If $u(z)$ is analytic in D, we can have a countable number of zeros accumulating near S. We want to use the fact that $u \in \mathcal{H}_p$ for some $p > 0$ to control the infinite product $\Pi_{i=1}^{\infty} f_{z_i}(z)$, that we may now have to deal with. Since $\log|u(z)|$ is subharmonic and we can assume that $u(0) \neq 0$

$$-\infty < c = \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta$$

for $r < 1$. If we take a finite number of zeros $z_1, \ldots, z_k$ and factor $u(z) = F_k(z)v_k(z)$ where $F_k(z) = \Pi_1^k f_{z_i}(z)$ is continuous on $D \cup S$ and $|F_k(z)| = 1$ on S, we get

$$\begin{aligned}
\log|v_k(0)| &\leq \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log|v_k(re^{i\theta})|d\theta \\
&= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta \\
&\leq \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta \\
&\leq C
\end{aligned}$$

uniformly in k. Since

$$\log|v_k(0)| = \log|u(0)| - \log|F_k(0)| = \log|u(0)| - \sum_{j=1}^{k} \log|z_j|$$

$$-\sum_{j=1}^{k} \log|z_j| \leq -\log|u(0)| + C$$

Denoting $C - c$ by C_1,

$$\sum(1 - |z_j|) \leq \sum -\log|z_j| \leq C_1$$

One sees from this that actually the infinite product $F(z) = \Pi_j f_{z_j}(z)e^{-i\,a_j}$ converges with proper phase factors a_j. We write $-z_j = |z_j|e^{-i\,a_j}$. Then

$$
\begin{aligned}
1 - f_{z_i}(z)e^{-i\,a_j} &= 1 + \frac{z - z_j}{1 - z\bar{z}_j}\frac{|z_j|}{z_j} \\
&= \frac{z_j - z|z_j|^2 + z|z_j| - z_j|z_j|}{z_j(1 - z\bar{z}_j)} \\
&= \frac{(1 - |z_j|)(z_j + z|z_j|)}{z_j(1 - z\bar{z}_j)}
\end{aligned}
$$

Therefore $|1 - f_{z_j}(z)e^{-i\,a_j}| \le C(1-|z_j|)(1-|z|)^{-1}$ and if we redefine $F_n(z)$ by

$$F_n(z) = \Pi_{j=1}^n f_{z_j}(z)e^{-i\,a_j}$$

we have the convergence

$$\lim_{n\to\infty} F_n(z) = F(z) = \Pi_{j=1}^\infty f_{z_j}(z)e^{-i\,a_j}$$

uniformly on compact subsets of D as $n \to \infty$. It follows from $|F_n(z)| \le 1$ on D that $|F(z)| \le 1$ on D. The functions $v_n(z) = \frac{u(z)}{F_n(z)}$ are analytic in D (as the only zeros of F_n are zeros of u) and are seen easily to converge to the limit $v = \frac{u}{F}$ so that $u = Fv$. Moreover $F_n(z)$ are continuous near S and $|F_n(z)| \equiv 1$ on S. Therefore,

$$
\begin{aligned}
\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |v_n(re^{i\,\theta})|^p d\theta &= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |v_n(re^{i\,\theta})|^p d\theta \\
&= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|u(re^{i\,\theta})|^p}{|F_n(re^{i\,\theta})|^p} d\theta \\
&= \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta
\end{aligned}
$$

Since $v_n(z) \to v(z)$ uniformly on compact subsets of D, by Fatou's lemma,

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |v(re^{i\,\theta})|^p d\theta \le \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\,\theta})|^p d\theta \tag{5.4}$$

In other words we have succeeded in writing $u = Fv$ with $|F(z)| \leq 1$, removing all the zeros of u, but v still satsfying (5.4). In order to complete the proof of the theorem it only remains to prove that $|F(z)| = 1$ a.e. on S. From (5.4) and the relation $u = vF$, it is not hard to see that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |v(re^{i\theta})|^p(1 - |F(re^{i\theta})|^p)d\theta = 0$$

Since $F(re^{i\theta})$ is known to have a boundary limit F^* to show that $|F^*| = 1$ a.e. all we need is to get a control on the Lebesgue measure of the set $\{\theta : |v(re^{i\theta})| \leq \delta\}$. It is clearly sufficient to get a bound on

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |\log|v(re^{i\theta})||d\theta$$

Since $\log^+ v$ can be dominated by $|v|^p$ with any $p > 0$, it is enough to get a lower bound on $\frac{1}{2\pi}\int_0^{2\pi} \log|v(re^{i\theta})|d\theta$ that is uniform as $r \to 1$. Clearly

$$\frac{1}{2\pi}\int_0^{2\pi} \log|v(re^{i\theta})|d\theta \geq \log|u(0)|$$

is sufficient. □

Theorem 5.4. *Suppose $u \in \mathcal{H}_p$ for some $p > 0$. Then $\lim_{r\to 1} u(re^{i\theta}) = u^*(e^{i\theta})$ exists in the following sense*

$$\lim_{r\to 1}\int_0^{2\pi} |u(re^{i\theta}) - u^*(e^{i\theta})|^p d\theta = 0$$

Moreover, if $p \geq 1$, u has the Poisson kernel representation in terms of u^*.

Proof. If $u \in \mathcal{H}_p$, according to Theorem 5.3, we can write $u = vF$ with $v \in \mathcal{H}_p$ which is zero free and $|F| \leq 1$. Choose an integer k such that $kp > 1$. Since v is zero free $v = w^k$ for some $w \in \mathcal{H}_{kp}$. Now $w(re^{i\theta})$ has a limit w^* in $L_{kp}(S)$. Since $|F| \leq 1$ and has a radial limit F^* it is clear the u has a limit $u^* \in L_p(S)$ given by $u^* = (w^*)^k F^*$. If $0 < p \leq 1$ to show convergence in the sense claimed above, we only have to prove the uniform integrability of $|u(re^{i\theta})|^p = |w(re^{i\theta})|^{kp}$ which follows from the convergence of w in $L_{kp}(S)$. If $p \geq 1$ it is easy to obtain the Poisson representation on S by taking the limit as $r \to 1$ from the representation on $|z| = r$ which is always valid. □

We can actually prove a better version of Theorem 5.3. Let $u \in \mathcal{H}_p$ for some $p > 0$, be arbitrary but not identically zero. We can start with the inequality

$$-\infty < \log|u(r_0 e^{i\theta_0})| \leq \frac{r^2 - r_0^2}{2\pi} \int_0^{2\pi} \frac{\log|u(re^{i(\theta_0 - \varphi)})|}{r^2 - 2rr_0\cos\varphi + r_0^2} d\varphi \qquad (5.5)$$

where $z_0 = r_0 e^{i\theta_0}$ is such that $r_0 = |z_0| < 1$ and $|u(z_0)| > 0$. We can use the uniform integrability of $\log^+|u(re^{i\theta})|$ as $r \to 1$, and conclude from Fatou's lemma that

$$\int_0^{2\pi} \frac{|\log|u(e^{i(\theta_0 - \varphi)})||}{1 - 2r_0\cos\varphi + r_0^2} d\varphi < \infty$$

Since the Poisson kernel is bounded above as well as below (away from zero) we conclude that the boundary function $u(e^{i\theta})$ satisfies

$$\int_0^{2\pi} |\log|u(e^{i\theta})||d\theta < \infty$$

We define $f(re^{i\theta})$ by the Poisson integral

$$f(re^{i\theta}) = \frac{1 - r^2}{4\pi} \int_0^{2\pi} \frac{\log|u(e^{i(\theta - \varphi)})|}{1 - 2r\cos\varphi + r^2} d\varphi$$

to be Harmonic with boundary value $\log|u(e^{i\theta})|$. From the inequality (5.5) it follows that $f(re^{i\theta}) \geq \log|u(re^{i\theta})|$. We can then take the conjugate harmonic function g so that $w(\cdot)$ given by $w(re^{i\theta}) = f(re^{i\theta}) + ig(re^{i\theta})$ is analytic. We define $v(z) = e^{w(z)}$ so that $\log|v| = f$. We can write $u = Fv$ that produces a factorization of u with a zero free v and F with $|F(z)| \leq 1$ on D. Since the boundaru values of $\log|u|$ and $\log|v|$ match on S, the boundary values of F which exist must satisfy $|F| = 1$ a.e. on S. We have therefore proved

Theorem 5.5. *Any u in $\mathcal{H}_p$, with $p > 0$, can be factored as $u = Fv$ with the following properties: $|F| \leq 1$ on D, $|F| = 1$ on S, v is zero free in D and $\log|v|$, which is harmonic in D, is given by the Poisson formula in terms of its boundary value $\log|v(e^{i\theta})| = \log|u(e^{i\theta})|$ which is in $L_1(S)$. Such a factorization is essentially unique, the only ambiguity being a mutiplicatve constant of absolute value 1.*

Remark. The improvement over Theorem 5.3 is that we have made sure that $\log|v|$ is not only Harmonic in D but actually takes on its boundary value in the sense of $L_1(S)$. This provides the uniqueness that was missing before. As an example consider the Poisson kernel itself.

$$u(z) = e^{\frac{z+1}{z-1}}$$

$|u(z)| < 1$ on D, $u(re^{i\theta}) \to e^{i\cot\frac{\theta}{2}}$ as $r \to 1$. Such a factor is without zeros and would be left alone in Theorem 5.3, but removed now.

There are characterizations of the factor u that occurs in $u = vF$.

Theorem 5.6. *Let $u \in \mathcal{H}_2$ be arbitrary and nontrivial. Then 1 belongs to the span of $\{z^n u : n \geq 0\}$ if and only if*

$$\log|u(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta})|\, d\theta \tag{5.6}$$

Proof. Let some linear span of $\{z^n u : n \geq 0\}$ converge to 1. In other words $\|p_n(z)u(z) - 1\|_{\mathcal{H}_2} \to 0$ for some polynomials $p_n(\cdot)$. Then

$$\log|p_n(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|p_n(e^{i\theta})|\, d\theta$$

and $\log|p_n(e^{i\theta})u(e^{i\theta})| \to 0$ as $n \to \infty$ in measure on S and $\log^+|p_n(e^{i\theta})u(e^{i\theta})|$ is uniformly integrable. Therefore by Fatou's lemma

$$0 \geq \limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log|p_n(e^{i\theta})u(e^{i\theta})|\, d\theta$$

It follows that

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log|p_n(e^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta})|\, d\theta \\
&\geq \limsup_{n\to\infty} \log|p_n(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta})|\, d\theta \\
&= -\log|u(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta})|\, d\theta
\end{aligned}$$

which implies that

$$\log|u(0)| \geq \frac{1}{2\pi}\int_0^{2\pi} \log|u(e^{i\theta})|\, d\theta$$

The reverse inequality is always valid and so we are done with one half.

To prove the converse we first establish two lemmas. Let us suppose that $u \in \mathcal{H}_2$ is not identically zero and let $\mathcal{K}$ be the span of uH as H varies over $\mathcal{H}_\infty$.

Lemma 5.7. *Pick a such that $|u(a)| > 0$ and take $k_a \in \mathcal{K}$ to be the orthogonal projection in $\mathcal{K}$ of $f_a(z) = \frac{1}{1-\bar{a}z}$. Than $|k_a(e^{i,\theta})|^2 = c(a)P_a(e^{i\theta})$ where P is the Poisson kernel and $c(a) > 0$ is a positive constant*

Proof. Note that by Cauchy's formula for any $v \in \mathcal{H}_2$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\overline{f_a(e^{i\theta})}v(e^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-ae^{-i\theta}}v(e^{i\theta})d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{e^{i\theta}-a}v(e^{i\theta})de^{i\theta} \\ &= \frac{1}{2\pi i}\int_C \frac{v(z)}{z-a}dz \\ &= v(a) \qquad (5.7)\end{aligned}$$

Note that $(f_a - k_a) \perp \mathcal{K}$. Writing the orthogonality relations in terms of the boundary values, and noting that $z^n k_a \in \mathcal{K}$ for $n \geq 0$,

$$\int_0^{2\pi}[\overline{f_a(e^{i\theta}) - k_a(e^{i\theta})}]e^{in\theta}k_a(e^{i\theta})d\theta =< f_a - k_a, z^n k_a >= 0 \qquad (5.8)$$

On the other hand for $n \geq 0$, since $z^n k_a \in \mathcal{H}_2$, by (5.7)

$$\int_0^{2\pi}\overline{f_a(e^{i\theta})}e^{in\theta}k_a(e^{i\theta})d\theta = 2\pi a^n k_a(a)$$

Combining with equation (5.8) we get for $n \geq 0$,

$$\int_0^{2\pi} e^{in\theta}|k_a(e^{i\theta})|^2 d\theta = 2\pi k_a(a)a^n$$

But $|k_a|^2$ is real and therefore $k_a(a)$ must be real and

$$\int_0^{2\pi} e^{in\theta}|k_a(e^{i\theta})|^2 d\theta = \begin{cases} 2\pi k_a(a)a^n & \text{if } n > 0 \\ 2\pi k_a(a) & \text{if } n = 0 \\ 2\pi k_a(a)\bar{a}^n & \text{if } n < 0 \end{cases}$$

This implies that $|k_a(e^{i\theta})|^2 \equiv c(a)P_a(e^{i\theta})$ on S where P_a is the Poisson kernel. If $c(a) = 0$, it follows that $k_a \equiv 0$ and hence $f_a \perp \mathcal{K}$. Since $u \in \mathcal{K}$, this implies by (5.7) that

$$< f_a, u >= 2\pi u(a) = 0$$

which is ruled out by the choice of a. □

Lemma 5.8. *The span of $\{k_a H\}$ as H varies over $\mathcal{H}_\infty$ is all of $\mathcal{K}$.*

Proof. If not, let $v \in \mathcal{K}$ be such that $v \perp k_a H$ for all $H \in \mathcal{H}_\infty$. We have then, for $n \geq 0$, taking $H = z^n$,

$$\int_0^{2\pi} \overline{k_a(e^{i\theta})} e^{-in\theta} v(e^{i\theta}) d\theta =< k_a z^n, v >= 0$$

For $n = -m < 0$, $z^m v \in \mathcal{K}$ and

$$\int_0^{2\pi} \overline{k_a(e^{i\theta})} e^{-in\theta} v(e^{i\theta}) d\theta =< k_a, z^m v >=< f_a, z^m v >= 2\pi a^m v(a)$$

Now Fourier inversion gives

$$\begin{aligned}\overline{k_a(e^{i\theta})} v(e^{i\theta}) &= v(a) \sum_{m=1}^{\infty} a^m e^{-im\theta} = v(a) \frac{ae^{-i\theta}}{1 - ae^{-i\theta}} \\ &= c_1(a) \frac{1}{e^{i\theta} - a} = c_2(a) P_a(e^{i\theta})(e^{-i\theta} - \bar{a})\end{aligned}$$

Multiplying by k_a and remembering that $|k_a|^2 = c(a)P_a$ with $c(a) > 0$, we obtain $c(a)v(e^{i\theta}) = c_2(a)k_a(e^{i\theta})(e^{-i\theta} - \bar{a})$. This leads to

$$v(e^{i\theta}) = c_4(a) \frac{k_a(e^{i\theta})}{e^{-i\theta} - \bar{a}} = c_4(a) \frac{k_a(e^{i\theta})e^{i\theta}}{1 - \bar{a}e^{i\theta}}$$

Therefore $v = k_a H$ with $H(z) = \frac{z}{1-\bar{a}z} \in \mathcal{H}_\infty$ contradicting $v \perp Hk_a$ for all $H \in \mathcal{H}_\infty$ and forcing v to be 0. □

We are now ready to prove the converse. If the span of $\{z^n u : n \geq 0\}$ is $\mathcal{K} \subset \mathcal{H}_2$ is a proper subspace, there is an a, f_a and k_a such that $u = k_a v$ for some $v \in \mathcal{H}_\infty$ and $|k|^2(e^{i\theta}) = c(a)P_a(e^{i\theta})$, the Poisson kernel for $a \in D$. For the Poisson kernel it is easy to verify that

$$\log |P_a(0)| < \frac{1}{2\pi} \int_0^{2\pi} \log |P_a(e^{i\theta})| \, d\theta$$

for any $a \in D$. Therefore we cannot have (5.6) satisfied. □

5.4 Connection to prediction theory.

Suppose $f(e^{i\theta}) \geq 0$ is a weight that is in $L_1(S)$. We consider the Hilbert Space $H = L_2(S, f)$ of functions u that are square integrable with respect to the weight f, i.e. g such that $\int_0^{2\pi} |g(e^{i\theta})|^2 f(e^{i\theta}) d\theta < \infty$. The trigonometric functions $\{e^{in\theta} : -\infty < n < \infty\}$ are still a basis for H, though they may not be mutually orthogonal. We define $H_k = \text{span}\{e^{i\,n\,\theta} : n \geq k\}$. It is clear the $H_k \supset H_{k+1}$ and mutiplication by $e^{\pm i\theta}$ is a unitary map $U^{\pm 1}$ of H onto itself that sends H_k onto $H_{k\pm 1}$. We are interested in calculating the orthogonal projection $e_1(e^{i\,\theta})$ of 1 into H_1 along with the residual error $\|1 - e_1(e^{i\,\theta})\|_2^2$. There are two possibilities. Either $1 \in H_1$ in which case $H_0 = H_1$ and hence $H_k = H$ for all k, or H_0 is spanned by H_1 and a unit vector $u_0 \in H_0$ that is orthogonal to H_1. If we define $u_k = U^k u_0$, then $H = \oplus_{j=-\infty}^{\infty} u_j \oplus H_\infty$ where $H_\infty = \cap_k H_k$. In a nice situation we expect that $H_{-\infty} = \{0\}$. However if $e^{i\,\theta} \in H_0$ as we saw $H_\infty = H$. If $f(e^{i\,\theta}) \equiv c$ then of course $u_k = e^{i\,k\,\theta}$.

Theorem 5.9. *Let us suppose that*

$$\int_0^{2\pi} \log f(e^{i\,\theta}) d\theta > -\infty \tag{5.9}$$

Then $H_\infty = \{0\}$ and the residual error is given by

$$\begin{aligned} \|1 - e_1(e^{i\,\theta})\|_2^2 &= \inf_{\{a_j\}} \|1 - \sum_{j \geq 1} a_j e^{i\,j\,\theta}\|_2^2 \\ &= 2\pi \exp[\frac{1}{2\pi} \int_0^{2\pi} \log f(e^{i\,\theta}) d\theta] > 0 \end{aligned} \tag{5.10}$$

Proof. We will split the proof into several steps. The prediction problem involves

$$\inf_{\{a_j\}} \|1 - \sum_{j \leq -1} a_j e^{i\,j\,\theta}\|_2^2$$

We seems to be predicting the past given the future. But the calculations are the same.

$$\begin{aligned} \|e_0(e^{i\,\theta}) - e^{-i\,\theta}\|_2^2 &= \inf_{\{a_j\}} \|e^{-i\,\theta} - \sum_{j \geq 0} a_j e^{i\,j\,\theta}\|_2^2 \\ &= 2\pi \exp[\frac{1}{2\pi} \int_0^{2\pi} \log f(e^{i\,\theta}) d\theta] > 0 \end{aligned} \tag{5.11}$$

Step 1. We write $f(e^{i\theta}) = |u(e^{i\theta})|^2$, where u is the boundary value of a function $u(re^{i\theta})$ in $\mathcal{H}_2$. Note that, if this were possible, then according to Theorem 5.3 one can assume with out loss of generality that $u(0) \neq 0$ and for $0 < r < 1$

$$-\infty < \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta$$

We can let $r \to 1$, use the domination of $\log^+|u|$ by $|u|$ and Fatou's lemma on $\log^-|u|$. We get

$$-\infty < \log|u(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta = \frac{1}{4\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

We see that the condition (5.9) is necessary for the representation that we seek. We begin with the function $\frac{1}{2}\log f \in L_1(S)$ and construct $F(re^{i\theta})$ given by the Poisson formula

$$F(re^{i\theta}) = \frac{1-r^2}{4\pi}\int_0^{2\pi} \frac{\log f(e^{i(\theta-\varphi)})}{1-2r\cos\varphi + r^2}d\varphi$$

to be Harmonic with boundary value $\frac{1}{2}\log f$. We then take the conjugate harmonic function G so that $w(\cdot)$ given by $w(re^{i\theta}) = F(re^{i\theta}) + iG(re^{i\theta})$ is analytic. We define $u(z) = e^{w(z)}$.

$$\begin{aligned}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta &= \int_0^{2\pi} \exp[2F(re^{i\theta})]d\theta \\ &\leq \int_0^{2\pi} \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(e^{i(\theta-\varphi)})}{1-2r\cos\varphi + r^2}d\varphi d\theta \\ &= \int_0^{2\pi} f(e^{i\theta})d\theta\end{aligned}$$

Therefore $u \in \mathcal{H}_2$ and $\lim_{r\to 1} u(re^{i\theta}) = u(e^{i\theta})$ exists in $L_2(S)$. Clearly

$$|u(e^{i\theta})| = \exp[\lim_{r\to 1} F(re^{i\theta})] = \sqrt{f(e^{i\theta})}$$

and $f = |u|^2$ on S. It is easily seen that $u(z) = \sum_{n\geq 0} a_n z^n$ with

$$\sum_{n\geq 0} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$$

Step 2. Our representation has the additional property that $u(z)$ is zero free in D and satifies (5.6). Suppose $h(re^{i\theta})$ is any function in $\mathcal{H}_2$ with boundary value $h(e^{i\theta})$ with $|h| = \sqrt{f}$ that also satsifies

$$\log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log f d\theta$$

then

$$\log|h(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta$$

By Fatou's lemma applied to $\log^-|h|$ as $r \to 1$ we get

$$\limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log f(e^{i\theta})d\theta$$

Therefore equality holds in Fatou's lemma implying the uniform integrabilty as well as the convergence in $L_1(S)$ of $\log|h(re^{i\theta})|$ to $\frac{1}{2}\log f(e^{i\theta})$ as $r \to 1$. In particular for $0 < r < 1$,

$$\log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|d\theta$$

and hence h is zero free in D. Consequently, for $0 \leq r < r' < 1$

$$\log|h(re^{i\theta})| = \frac{r'^2 - r^2}{2\pi}\int_0^{2\pi} \frac{\log|h(re^{i(\theta-\varphi)})|}{r'^2 - 2r'r\cos d\varphi + r^2}d\varphi$$

We can let $r' \to 1$ use the convergence of $\log|h(re^{i\theta})|$ to $\frac{1}{2}\log f$ in $L_1(S)$ to conclude

$$\log|h(re^{i\theta})| = \frac{1-r^2}{4\pi}\int_0^{2\pi} \frac{\log f(e^{i(\theta-\varphi)})}{1 - 2r\cos d\varphi + r^2}d\varphi$$

Therfore the representation of $f(e^{i\theta}) = |u(e^{i\theta})|^2$, with $u(e^{i\theta})$ the boundary value of $u \in \mathcal{H}_2$ that satisfies condition (5.6) is unique to within a multiplicative constant of absolute value 1. The significance of making the choice of u so that the condition (5.6) is valid, is that we can conclude that $\{z^j u(z)\}$ spans all of $\mathcal{H}_2$. $\square$